coachingactuaries

You have what it takes to pass

Exam P

updated 08/23/19

GENERAL PROBABILITY

Basic Probability Relationships

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A \cup B \cup C)$

$= \Pr(A) + \Pr(B) + \Pr(C)$

$\quad - \Pr(A \cap B) - \Pr(B \cap C) - \Pr(A \cap C)$

$\quad + \Pr(A \cap B \cap C)$

$\Pr(A') = 1 - \Pr(A)$

Law of Total Probability

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap A_i)$$

De Morgan's Law

$\Pr[(A \cup B)'] = \Pr(A' \cap B')$

$\Pr[(A \cap B)'] = \Pr(A' \cup B')$

Conditional Probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Independence

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$\Pr(A|B) = \Pr(A)$

Bayes' Theorem

$$\Pr(A_k|B) = \frac{\Pr(B|A_k) \cdot \Pr(A_k)}{\sum_{i=1}^{n} \Pr(B|A_i) \cdot \Pr(A_i)}$$

Combinatorics

$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

$${}_nP_k = \frac{n!}{(n-k)!}$$

$${}_nC_k = \binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

UNIVARIATE PROBABILITY DISTRIBUTIONS

**Learn both discrete and continuous cases*

***Probability Mass Function (PMF)**

$\sum_{\text{all } x} p_X(x) = 1$

$\Pr(X = a) = 0$ (continuous)

***Cumulative Distribution Function (CDF)**

$F_X(x) = \Pr(X \le x) = \sum_{i \le x} p_X(i)$

$\Pr(a < X \le b) = F_X(b) - F_X(a)$

$f_X(x) = \frac{d}{dx} F_X(x)$ (continuous)

***Expected Value**

$E[c] = c$

$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$

$E[g(X)] = \int_0^{\infty} g'(x) \cdot S_X(x)\, dx$, for $x \ge 0$

$$E[g(X)|j \le X \le k] = \frac{\int_j^k g(x) \cdot f_X(x)\, dx}{\Pr(j \le X \le k)}$$

$E[c \cdot g(X)] = c \cdot E[g(X)]$

$E[g_1(X) + \cdots + g_k(X)]$

$\quad = E[g_1(X)] + \cdots + E[g_k(X)]$

Variance, Standard Deviation, and Coefficient of Variation

$Var[X] = E[X^2] - (E[X])^2$

$Var[aX + b] = a^2 \cdot Var[X]$

$Var[c] = 0$

$SD[X] = \sqrt{Var[X]}$

$CV[X] = SD[X]/E[X]$

***Moment Generating Function (MGF)**

$M_X(t) = E[e^{tX}]$

$M_{aX+b}(t) = e^{bt} \cdot M_X(at)$

$M_X(0) = 1$

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ (independent)

$$\frac{d^n}{dt^n} M_X(t)\bigg|_{t=0} = E[X^n]$$

Probability Generating Function (PGF)

$P_X(t) = E[t^X]$

$P_X(0) = p_X(0)$

$$\frac{\frac{d^n}{dt^n} P_X(t)\Big|_{t=0}}{n!} = p_X(n)$$

$$\frac{d^n}{dt^n} P_X(t)\bigg|_{t=1} = E[X(X-1)\ldots(X-n+1)]$$

Percentiles

The $100p^{\text{th}}$ percentile is the smallest value of π_p where $F_X(\pi_p) \ge p$.

Univariate Transformation

$$f_Y(y) = f_X[g^{-1}(y)] \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$$

where $y = g(x) \Leftrightarrow x = g^{-1}(y)$

Discrete Distributions						
	PMF	**Mean**	**Variance**	**MGF**	**PGF**	**Special Properties**
Discrete Uniform	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$	$\frac{e^{at}-e^{(b+1)t}}{(1-e^t)(b-a+1)}$	–	–
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$(pe^t+q)^n$	$(pt+q)^n$	–
Hypergeometric	$\left[\binom{m}{x}\cdot\binom{N-m}{n-x}\right]/\binom{N}{n}$	$n\cdot\frac{m}{N}$	–	–	–	–
Geometric	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{pt}{1-(1-p)t}$	Memoryless property
X: trials; Y: failures $X=Y+1$	$(1-p)^y p$	$\frac{1}{p}-1$		$\frac{p}{1-(1-p)e^t}$	$\frac{p}{1-(1-p)t}$	
Negative Binomial	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$	$\frac{r}{p}$	$r\left(\frac{1-p}{p^2}\right)$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$	$\left(\frac{pt}{1-(1-p)t}\right)^r$	Neg Bin$(r=1,p)\sim$ Geometric(p)
X: trials; Y: failures $X=Y+r$	$\binom{y+r-1}{r-1}p^r(1-p)^y$	$\frac{r}{p}-r$		$\left(\frac{p}{1-(1-p)e^t}\right)^r$	$\left(\frac{p}{1-(1-p)t}\right)^r$	
Poisson	$\frac{e^{-\lambda}\cdot\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$	$e^{\lambda(t-1)}$	Sum of independent Poissons ~ Poisson $(\lambda=\sum_{i=1}^n\lambda_i)$

Continuous Distributions						
	PDF	**CDF**	**Mean**	**Variance**	**MGF**	**Special Properties**
Continuous Uniform	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{t(b-a)}$	$(X\mid X>c)\sim$ Uniform(c,b) $(X-c\mid X>c)\sim$ Uniform$(0,b-c)$
Exponential	$\frac{1}{\theta}e^{-\frac{x}{\theta}}$	$1-e^{-\frac{x}{\theta}}$	θ	θ^2	$\frac{1}{1-\theta t}$	Memoryless property: $(X-a\mid X>a)\sim X$
Gamma	$\frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\theta^\alpha}\cdot e^{-\frac{x}{\theta}}$	$1-\sum_{k=0}^{\alpha-1}\Pr(Y=k)$, $Y\sim$ Poisson$\left(\lambda=\frac{x}{\theta}\right)$	$\alpha\theta$	$\alpha\theta^2$	$\left(\frac{1}{1-\theta t}\right)^\alpha$	Sum of α independent exponentials$(\theta)\sim$ Gamma(α,θ)
Normal	$\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$Z=\frac{X-\mu}{\sigma}$ $\Pr(Z\le z)=\Phi(z)$	μ	σ^2	$e^{\mu t+\frac{\sigma^2t^2}{2}}$	Symmetry: $\Pr(Z\le z)=\Pr(Z\ge -z)$ $\Pr(Z\le -z)=\Pr(Z\ge z)$ Sum of independent normals ~ Normal$\left(\mu=\sum_{i=1}^n\mu_i,\sigma^2=\sum_{i=1}^n\sigma_i^2\right)$

MULTIVARIATE PROBABILITY DISTRIBUTIONS

**Learn both discrete and continuous cases*

***Joint PMF and CDF**

$\sum_{\text{all } x} \sum_{\text{all } y} p_{X,Y}(x,y) = 1$

$F_{X,Y}(x,y) = \sum_{s \le x} \sum_{t \le y} p_{X,Y}(s,t)$

$\frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x,y) = f_{X,Y}(x,y)$ (continuous)

***Marginal Distributions and Conditional Distributions**

$p_X(x) = \sum_{\text{all } y} p_{X,Y}(x,y)$

$p_Y(y) = \sum_{\text{all } x} p_{X,Y}(x,y)$

$p_{X|Y}(x|Y=y) = p_{X,Y}(x,y)/p_Y(y)$

***Joint Expected Value and Conditional Expectation**

$E[g(X,Y)]$

$\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)\, dy\, dx$

$E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|Y=y)\, dx$

Double Expectation and Law of Total Variance

$E[X] = E\big[E[X|Y]\big]$

$Var[X] = E\big[Var[X|Y]\big] + Var\big[E[X|Y]\big]$

Covariance and Correlation Coefficient

$Cov[X,Y] = E[XY] - E[X]E[Y]$

$Cov[aX,bY] = ab \cdot Cov[X,Y]$

$Cov[X,X] = Var[X]$

$Var[aX+bY] = a^2 Var[X] + b^2 Var[Y]$
$\qquad\qquad + 2ab \cdot Cov[X,Y]$

$$\rho_{X,Y} = Corr[X,Y] = \frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}}$$

Independence

$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$

$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$E[h(X) \cdot k(Y)] = E[h(X)] \cdot E[k(Y)]$

$M_{X,Y}(s,t) = M_X(s) \cdot M_Y(t)$

$Cov[X,Y] = 0$

$\rho_{X,Y} = 0$

***Joint MGF**

$M_{X,Y}(s,t) = E[e^{sX+tY}]$

$E[X] = \frac{\partial}{\partial s} M_{X,Y}(s,t)\Big|_{s=t=0}$

$E[Y] = \frac{\partial}{\partial t} M_{X,Y}(s,t)\Big|_{s=t=0}$

$E[X^m Y^n] = \frac{\partial^{m+n}}{\partial s^m \partial t^n} M_{X,Y}(s,t)\Big|_{s=t=0}$

$M_{X,Y}(t,t) = M_{X+Y}(t)$

Multivariate Transformation

$f_{W_1,W_2}(w_1,w_2)$

$= f_{X_1,X_2}[h_1(w_1,w_2), h_2(w_1,w_2)] \cdot |J|$

where

$x_1 = h_1(w_1,w_2)$

$x_2 = h_2(w_1,w_2)$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{vmatrix}$$

Multinomial Distribution

$\Pr(X_1 = x_1, \ldots, X_k = x_k)$

$\quad = \frac{n!}{x_1! \cdot \ldots \cdot x_k!} \cdot p_1^{x_1} \cdot \ldots \cdot p_k^{x_k}$

$E[X_i] = np_i$

$Var[X_i] = np_i(1-p_i)$

$Cov[X_i, X_j] = -np_i p_j, \qquad i \ne j$

Bivariate Continuous Uniform

$f_{X,Y}(x,y) = \frac{1}{\text{Area of domain}}$

$\Pr(\text{region}) = \frac{\text{Area of region}}{\text{Area of domain}}$

Bivariate Normal

For $X \sim \text{Normal}(\mu_X, \sigma_X^2)$ and $Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$,

$(Y|X=x) \sim \text{Normal}$

where

$E[Y|X=x] = \mu_Y + \rho \cdot \sigma_Y \left(\frac{x-\mu_X}{\sigma_X}\right)$

$Var[Y|X=x] = \sigma_Y^2(1-\rho^2)$

Expectation and Variance for Sum and Average of I.I.D. Random Variables

$S = X_1 + \cdots + X_n$

$\bar{X} = [X_1 + \cdots + X_n]/n$

$E[S] = n \cdot E[X_i]$

$E[\bar{X}] = E[X_i]$

$Var[S] = n \cdot Var[X_i]$

$Var[\bar{X}] = (1/n) \cdot Var[X_i]$

Central Limit Theorem

The sum of a large number of identically and independently distributed (i.i.d.) random variables approximately follows a normal distribution.

Order Statistics

$X_{(1)} = \min(X_1, X_2, \ldots, X_n)$

$X_{(n)} = \max(X_1, X_2, \ldots, X_n)$

For i.i.d. random variables,

$S_{X_{(1)}}(x) = [S_X(x)]^n$

$F_{X_{(n)}}(x) = [F_X(x)]^n$

Category	Definition of Payment, Y	$E[Y]$	
Deductible	$Y = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases}$	$\int_d^{\infty} (x-d) \cdot f_X(x)\, dx$ $\int_d^{\infty} S_X(x)\, dx$	For exponential: $\theta \cdot \Pr(X > d)$
Policy Limit	$Y = \begin{cases} X, & X < u \\ u, & X \ge u \end{cases}$	$\int_0^{u} x \cdot f_X(x)\, dx + u \cdot S_X(u)$ $\int_0^{u} S_X(x)\, dx$	For exponential: $\theta \cdot \Pr(X < u)$
Deductible and Policy Limit (u is the policy limit/ maximum payment)	$Y = \begin{cases} 0, & X \le d \\ X - d, & d < X < d + u \\ u, & X \ge d + u \end{cases}$	$\int_d^{d+u} (x-d) \cdot f_X(x)\, dx + u \cdot S_X(d+u)$ $\int_d^{d+u} S_X(x)\, dx$	For exponential: $\theta \cdot \Pr(d < X < d + u)$

Unreimbursed Loss, Z

If X is the loss and Y is the payment (i.e. reimbursed loss), then $X = Y + Z \Rightarrow Z = X - Y$, and $E[Z] = E[X] - E[Y]$.